On some properties of moduli of smoothness with Jacobi weights *

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Dedicated to the memory of our friend, colleague and collaborator Yingkang Hu (July 6, 1949 – March 11, 2016)

Abstract

We discuss some properties of the moduli of smoothness with Jacobi weights that we have recently introduced and that are defined as

$$\omega^{\varphi}_{k,r}(f^{(r)},t)_{\alpha,\beta,p} := \sup_{0\le h\le t} \left\| \mathcal{W}^{r/2+\alpha,r/2+\beta}_{kh}(\cdot)\Delta^k_{h\varphi(\cdot)}(f^{(r)},\cdot)\right\|_p$$

where $\varphi(x)=\sqrt{1-x^2}$, $\Delta^k_h(f,x)$ is the kth symmetric difference of f on $[-1,1]$,

$$\mathcal{W}^{\xi,\zeta}_{\delta}(x) := (1-x-\delta\varphi(x)/2)^{\xi}(1+x-\delta\varphi(x)/2)^{\zeta},$$

and $\alpha,\beta>-1/p$ if $0<p<\infty$, and $\alpha,\beta\ge 0$ if $p=\infty$.

We show, among other things, that for all $m,n\in\mathbb{N}$, $0<p\le\infty$, polynomials P_n of degree $<n$ and sufficiently small t,

$$\omega^{\varphi}_{m,0}(P_n,t)_{\alpha,\beta,p} \sim t\omega^{\varphi}_{m-1,1}(P_n',t)_{\alpha,\beta,p} \sim \cdots \sim t^{m-1}\omega^{\varphi}_{1,m-1}(P_n^{(m-1)},t)_{\alpha,\beta,p} \sim t^m \left\| w_{\alpha,\beta}\varphi^m P_n^{(m)}\right\|_p,$$

where $w_{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$ is the usual Jacobi weight.

In the spirit of Yingkang Hu's work, we apply this to characterize the behavior of the polynomials of best approximation of a function in a Jacobi weighted L_p space, $0<p\le\infty$. Finally we discuss sharp Marchaud and Jackson type inequalities in the case $1<p<\infty$.

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1 Introduction

Recall that the Jacobi weights are defined as $w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$, where parameters α and β are usually assumed to be such that $w_{\alpha,\beta} \in L_p[-1,1]$, *i.e.*,

$$\alpha, \beta \in J_p := \begin{cases} (-1/p, \infty), & \text{if } 0 < p < \infty, \\ [0, \infty), & \text{if } p = \infty. \end{cases}$$

We denote by $\mathbb{P}_n$ the set of all algebraic polynomials of degree $\le n-1$, and $L_p^{\alpha,\beta}(I) := \left\{ f \mid \|w_{\alpha,\beta} f\|_{L_p(I)} < \infty \right\}$, where $I \subseteq [-1,1]$. For convenience, if $I = [-1,1]$ then we omit I from the notation. For example, $\|\cdot\|_p := \|\cdot\|_{L_p[-1,1]}$, $L_p^{\alpha,\beta} := L_p^{\alpha,\beta}[-1,1]$, etc.

Following [5] we denote $\mathbb{B}_p^0(w_{\alpha,\beta}) := L_p^{\alpha,\beta}$, and

$$\mathbb{B}_p^r(w_{\alpha,\beta}) := \left\{ f \mid f^{(r-1)} \in AC_{loc} \quad \text{and} \quad \varphi^r f^{(r)} \in L_p^{\alpha,\beta} \right\}, \quad r \ge 1,$$

where AC_{loc} denotes the set of functions which are locally absolutely continuous in $(-1,1)$, and $\varphi(x) := \sqrt{1-x^2}$. Also (see [5]), for $k, r \in \mathbb{N}$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, let

$$\begin{aligned} (1.1) \qquad \omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} &:= \sup_{0 \le h \le t} \left\| \mathcal{W}_{kh}^{r/2+\alpha, r/2+\beta}(\cdot) \Delta_{h\varphi(\cdot)}^k (f^{(r)}, \cdot) \right\|_p \\ &= \sup_{0 < h \le t} \left\| \mathcal{W}_{kh}^{r/2+\alpha, r/2+\beta}(\cdot) \Delta_{h\varphi(\cdot)}^k (f^{(r)}, \cdot) \right\|_{L_p(\mathfrak{D}_{kh})}, \end{aligned}$$

where

$$\Delta_h^k(f, x; S) := \begin{cases} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(x - \frac{kh}{2} + ih), & \text{if } [x - \frac{kh}{2}, x + \frac{kh}{2}] \subseteq S, \\ 0, & \text{otherwise,} \end{cases}$$

is the kth symmetric difference, $\Delta_h^k(f,x) := \Delta_h^k(f, x; [-1,1])$,

$$\mathcal{W}_\delta^{\xi,\zeta}(x) := (1 - x - \delta\varphi(x)/2)^\xi (1 + x - \delta\varphi(x)/2)^\zeta,$$

and

$$\mathfrak{D}_\delta := [-1 + \mu(\delta), 1 - \mu(\delta)], \quad \mu(\delta) := 2\delta^2/(4+\delta^2)$$

(note that $\Delta_{h\varphi(x)}^k(f,x) = 0$ if $x \notin \mathfrak{D}_{kh}$).

We define the main part weighted modulus of smoothness as

$$(1.2) \qquad \Omega_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p} := \sup_{0 \le h \le t} \left\| w_{\alpha,\beta}(\cdot) \varphi^r(\cdot) \Delta_{h\varphi(\cdot)}^k (f^{(r)}, \cdot; \mathcal{I}_{A,h}) \right\|_{L_p(\mathcal{I}_{A,h})},$$

where $\mathcal{I}_{A,h} := [-1 + Ah^2, 1 - Ah^2]$ and $A > 0$.

We also denote

$$(1.3) \qquad \Psi_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} := \sup_{0 \le h \le t} \left\| w_{\alpha,\beta}(\cdot) \varphi^r(\cdot) \Delta_{h\varphi(\cdot)}^k (f^{(r)}, \cdot) \right\|_p,$$

i.e., $\Psi_{k,r}^\varphi$ is "the main part modulus $\Omega_{k,r}^\varphi$ with $A = 0$". However, we want to emphasize that while $\Omega_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p}$ with $A > 0$ and $\omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p}$ are bounded for all $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ (see [5, Lemma 2.4]), modulus $\Psi_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p}$ may be infinite for such functions (for example, this is the case for f such that $f^{(r)}(x) = (1-x)^{-\gamma}$ with $1/p \le \gamma < \alpha + r/2 + 1/p$).

Remark 1.1. *We note that the main part modulus is sometimes defined with the difference inside the norm not restricted to* $\mathfrak{I}_{A,h}$, i.e.,

$$\widetilde{\Omega}^{\varphi}_{k,r}(f^{(r)}, A, t)_{\alpha,\beta,p} := \sup_{0\le h\le t} \left\| w_{\alpha,\beta}(\cdot)\varphi^r(\cdot)\Delta^k_{h\varphi(\cdot)}(f^{(r)}, \cdot) \right\|_{L_p(\mathfrak{I}_{A,h})}. \tag{1.4}$$

Clearly, $\Omega^{\varphi}_{k,r}(f^{(r)}, A, t)_{\alpha,\beta,p} \le \widetilde{\Omega}^{\varphi}_{k,r}(f^{(r)}, A, t)_{\alpha,\beta,p}$. *Moreover, we have an estimate in the opposite direction as well if we replace* A *with a larger constant* A'. *For example,* $\widetilde{\Omega}^{\varphi}_{k,r}(f^{(r)}, A', t)_{\alpha,\beta,p} \le \Omega^{\varphi}_{k,r}(f^{(r)}, A, t)_{\alpha,\beta,p}$, *where* $A' = 2\max\{A, k^2\}$ *(see* (2.9)*). At the same time, if* A *is so small that* $\mathfrak{D}_{kh} \subset \mathfrak{I}_{A,h}$ *(for example, if* $A \le k^2/4$*), then* $\widetilde{\Omega}^{\varphi}_{k,r}(f^{(r)}, A, t)_{\alpha,\beta,p} = \Psi^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p}$. *Hence, all our results in this paper are valid with the modulus* (1.2) *replaced by* (1.4) *with an additional assumption that* A *is sufficiently large (assuming that* $A \ge 2k^2$ *will do).*

Throughout this paper, we use the notation

$$q := \min\{1, p\},$$

and $\boldsymbol{\varrho}$ stands for some sufficiently small positive constant depending only on α, β, k and q, and independent of n, to be prescribed in the proof of Theorem 2.1.

2 The main result

The following theorem is our main result.

Theorem 2.1. *Let* $k, n \in \mathbb{N}$, $r \in \mathbb{N}_0$, $A > 0$, $0 < p \le \infty$, $\alpha + r/2, \beta + r/2 \in J_p$, *and let* $0 < t \le \boldsymbol{\varrho} n^{-1}$, *where* $\boldsymbol{\varrho}$ *is some positive constant that depends only on* α, β, k *and* q. *Then, for any* $P_n \in \mathbb{P}_n$,

$$\begin{aligned}\omega^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} &\sim \Psi^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} \sim \Omega^{\varphi}_{k,r}(P_n^{(r)}, A, t)_{\alpha,\beta,p} \\ &\sim t^k \left\| w_{\alpha,\beta}\varphi^{k+r} P_n^{(k+r)} \right\|_p,\end{aligned} \tag{2.1}$$

where the equivalence constants depend only on k, r, α, β, A *and* q.

The following is an immediate corollary of Theorem 2.1 by virtue of the fact that, if $\alpha, \beta \in J_p$, then $\alpha + r/2, \beta + r/2 \in J_p$ for all $r \ge 0$.

Corollary 2.2. *Let* $m, n \in \mathbb{N}$, $A > 0$, $0 < p \le \infty$, $\alpha, \beta \in J_p$, *and let* $0 < t \le \boldsymbol{\varrho} n^{-1}$. *Then, for any* $P_n \in \mathbb{P}_n$, *and any* $k \in \mathbb{N}$ *and* $r \in \mathbb{N}_0$ *such that* $k + r = m$,

$$\begin{aligned}t^{-k}\omega^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} &\sim t^{-k}\Psi^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} \sim t^{-k}\Omega^{\varphi}_{k,r}(P_n^{(r)}, A, t)_{\alpha,\beta,p} \\ &\sim \left\| w_{\alpha,\beta}\varphi^{m} P_n^{(m)} \right\|_p,\end{aligned}$$

where the equivalence constants depend only on m, α, β, A *and* q.

It was shown in [5, Corollary 1.9] that, for $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2 + \alpha \ge 0$, $r/2 + \beta \ge 0$, $1 \le p \le \infty$, $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$, $\lambda \ge 1$ and all $t > 0$,

$$\omega^{\varphi}_{k,r}(f^{(r)}, \lambda t)_{\alpha,\beta,p} \le c\lambda^k \omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p}.$$

Hence, in the case $1 \le p \le \infty$, we can strengthen Corollary 2.2 for the moduli $\omega^{\varphi}_{k,r}$. Namely, the following result is valid.

Corollary 2.3. *Let $m, n \in \mathbb{N}$, $1 \le p \le \infty$, $\alpha, \beta \in J_p$, $\Lambda > 0$ and let $0 < t \le \Lambda n^{-1}$. Then, for any $P_n \in \mathbb{P}_n$, and any $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$ such that $k + r = m$,*

$$t^{-k} \omega_{k,r}^{\varphi}(P_n^{(r)}, t)_{\alpha,\beta,p} \sim \left\| w_{\alpha,\beta} \varphi^m P_n^{(m)} \right\|_p,$$

where the equivalence constants depend only on m, α, β and Λ.

Remark 2.4. *In the case $1 \le p \le \infty$, several equivalences in Theorem 2.1 and Corollary 2.2 follow from* [4, Theorems 4 and 5]*, since, as was shown in* [5, (1.8)]*, for $1 \le p \le \infty$,*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \sim \omega_{\varphi}^{k}(f^{(r)}, t)_{w_{\alpha,\beta}\varphi^r, p}, \quad 0 < t \le t_0, \tag{2.2}$$

where $\omega_\varphi^k(g,t)_{w,p}$ is the three-part weighted Ditzian-Totik modulus of smoothness (see e.g. [5, (5.1)] *for its definition).*

Note that it is still an open problem if (2.2) *is valid if* $0 < p < 1$.

Proof of Theorem 2.1. The main idea of the proof is not much different from that of [4, Theorems 3-5].

First, we note that it suffices to prove Theorem 2.1 in the case $r = 0$. Indeed, suppose we proved that, for $k, n \in \mathbb{N}$, $A > 0$, $0 < t \le \varrho n^{-1}$, $0 < p \le \infty$, $\alpha, \beta \in J_p$ and any polynomial $Q_n \in \mathbb{P}_n$,

$$\begin{aligned} \omega_{k,0}^{\varphi}(Q_n, t)_{\alpha,\beta,p} &\sim \Psi_{k,0}^{\varphi}(Q_n, t)_{\alpha,\beta,p} \sim \Omega_{k,0}^{\varphi}(Q_n, A, t)_{\alpha,\beta,p} \\ &\sim t^k \left\| w_{\alpha,\beta} \varphi^k Q_n^{(k)} \right\|_p. \end{aligned} \tag{2.3}$$

Then, if P_n is an arbitrary polynomial from $\mathbb{P}_n$, and r is an arbitrary natural number, assuming that $n > r$ (otherwise, $P_n^{(r)} \equiv 0$ and there is nothing to prove) and denoting $Q := P_n^{(r)} \in \mathbb{P}_{n-r}$, we have

$$\omega_{k,r}^{\varphi}(P_n^{(r)}, t)_{\alpha,\beta,p} = \omega_{k,0}^{\varphi}(Q, t)_{\alpha+r/2,\beta+r/2,p},$$

$$\Psi_{k,r}^{\varphi}(P_n^{(r)}, t)_{\alpha,\beta,p} = \Psi_{k,0}^{\varphi}(Q, t)_{\alpha+r/2,\beta+r/2,p},$$

$$\Omega_{k,r}^{\varphi}(P_n^{(r)}, t)_{\alpha,\beta,p} = \Omega_{k,0}^{\varphi}(Q, A, t)_{\alpha+r/2,\beta+r/2,p}$$

and

$$\left\| w_{\alpha,\beta} \varphi^{k+r} P_n^{(k+r)} \right\|_p = \left\| \omega_{\alpha+r/2,\beta+r/2} \varphi^k Q^{(k)} \right\|_p,$$

and so (2.1) follows from (2.3) with α and β replaced by $\alpha + r/2$ and $\beta + r/2$, respectively.

Now, note that it immediately follows from the definition that

$$\omega_{k,0}^{\varphi}(g, t)_{\alpha,\beta,p} \le \Psi_{k,0}^{\varphi}(g, t)_{\alpha,\beta,p}.$$

Also, for $A > 0$,

$$\Omega_{k,0}^{\varphi}(g, A, t)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(g, t)_{\alpha,\beta,p},$$

since $w_{\alpha,\beta}(x) \le c\mathcal{W}_{kh}^{\alpha,\beta}(x)$ for x such that $x \pm kh\varphi(x)/2 \in \mathfrak{I}_{A,h}$.

Hence, in order to prove (2.3), it suffices to show that

$$\Psi_{k,0}^{\varphi}(Q_n, t)_{\alpha,\beta,p} \le ct^k \left\| w_{\alpha,\beta} \varphi^k Q_n^{(k)} \right\|_p \tag{2.4}$$

and

$$t^k \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p \le c\Omega_{k,0}^{\varphi}(Q_n, A, t)_{\alpha,\beta,p}. \tag{2.5}$$

Recall the following Bernstein-Dzyadyk-type inequality that follows from [4, (2.24)]: if $0 < p \le \infty$, $\alpha, \beta \in J_p$ and $P_n \in \mathbb{P}_n$, then

$$\left\| w_{\alpha,\beta}\varphi^s P_n' \right\|_p \le cns \left\| w_{\alpha,\beta}\varphi^{s-1} P_n \right\|_p, \quad 1 \le s \le n-1,$$

where c depends only on α, β and q, and is independent of n and s.

This implies that, for any $Q_n \in \mathbb{P}_n$ and $k, j \in \mathbb{N}$,

$$\left\| w_{\alpha,\beta}\varphi^{k+j} Q_n^{(k+j)} \right\|_p \le (c_0 n)^j \frac{(k+j)!}{k!} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p, \quad 1 \le k+j \le n-1. \tag{2.6}$$

We now use the following identity (see [4, (2.4)]):

for any $Q_n \in \mathbb{P}_n$ and $k \in \mathbb{N}$, we have

$$\Delta_{h\varphi(x)}^k(Q_n, x) = \sum_{i=0}^{K} \frac{1}{(2i)!} \varphi^{k+2i}(x) Q_n^{(k+2i)}(x) h^{k+2i} \xi_{k+2i}^{2i}, \tag{2.7}$$

where $K := \lfloor (n-1-k)/2 \rfloor$, and $\xi_j \in (-k/2, k/2)$ depends only on k and j.

Applying (2.6), we obtain, for $0 \le i \le K$ and $0 < h \le t \le \varrho n^{-1}$,

$$\begin{aligned}
\left\| \frac{1}{(2i)!} w_{\alpha,\beta}\varphi^{k+2i} Q_n^{(k+2i)} \right\|_p h^{2i} |\xi_{k+2i}|^{2i} &\le (c_0 \varrho k/2)^{2i} \frac{(k+2i)!}{(2i)!k!} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p \\
&\le [c_0 \varrho k(k+1)/2]^{2i} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p \\
&\le B^{2i} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p,
\end{aligned}$$

where we used the estimate $(k+2i)!/((2i)!k!) \le (k+1)^{2i}$, and where ϱ is taken so small that the last estimate holds with $B := (1/3)^{1/(2q)}$. Note that $\sum_{i=1}^{\infty} B^{2iq} = 1/2$.

Hence, it follows from (2.7) that

$$\begin{aligned}
\left\| w_{\alpha,\beta}\Delta_{h\varphi}^k(Q_n, \cdot) \right\|_p^q &\le h^{kq} \sum_{i=0}^{K} \left\| \frac{1}{(2i)!} w_{\alpha,\beta}\varphi^{k+2i} Q_n^{(k+2i)} \right\|_p^q h^{2iq} |\xi|_{k+2i}^{2iq} \\
&\le h^{kq} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p^q \left(1 + \sum_{i=1}^{K} B^{2iq} \right) \\
&\le 3/2 \cdot h^{kq} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p^q.
\end{aligned}$$

This immediately implies

$$\Psi_{k,0}^{\varphi}(Q_n, t)_{\alpha,\beta,p} \le (3/2)^{1/q} t^k \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p,$$

and so (2.4) is proved.

Recall now the following Remez-type inequality (see *e.g.* [4, (2.22)]):

If $0 < p \le \infty$, $\alpha, \beta \in J_p$, $a \ge 0$, $n \in \mathbb{N}$ is such that $n > \sqrt{a}$, and $P_n \in \mathbb{P}_n$, then

$$\|w_{\alpha,\beta} P_n\|_p \le c \, \|w_{\alpha,\beta} P_n\|_{L_p[-1+an^{-2}, 1-an^{-2}]}, \tag{2.8}$$

where c depends only on α, β, a and q.

Note that

$$\begin{aligned}
\Omega^{\varphi}_{k,0}(Q_n, A, t)_{\alpha,\beta,p} &= \sup_{0\le h\le t} \left\| w_{\alpha,\beta}(\cdot) \Delta^k_{h\varphi(\cdot)}(Q_n, \cdot; \mathcal{I}_{A,h}) \right\|_{L_p(\mathcal{I}_{A,h})} \\
&= \sup_{0\le h\le t} \left\| w_{\alpha,\beta}(\cdot) \Delta^k_{h\varphi(\cdot)}(Q_n, \cdot) \right\|_{L_p(\mathcal{S}_{k,A,h})},
\end{aligned}$$

where the set $\mathcal{S}_{k,A,h}$ is an interval containing all x so that $x \pm kh\varphi(x)/2 \in \mathcal{I}_{A,h}$. Observe that

$$\mathcal{S}_{k,A,h} \supset \mathcal{I}_{A',h},$$

where $A' := 2\max\{A, k^2\}$, and so

$$\Omega^{\varphi}_{k,0}(Q_n, A, t)_{\alpha,\beta,p} \ge \sup_{0\le h\le t} \left\| w_{\alpha,\beta}(\cdot) \Delta^k_{h\varphi(\cdot)}(Q_n, \cdot) \right\|_{L_p(\mathcal{I}_{A',h})}. \tag{2.9}$$

Now it follows from (2.7) that $\Delta^k_{h\varphi(x)}(Q_n, x)$ is a polynomial from $\mathbb{P}_n$ if k is even, and it is a polynomial from $\mathbb{P}_{n-1}$ multiplied by φ if k is odd.

Hence, (2.8) implies that, for $h \le 1/(\sqrt{2A'}n)$,

$$\begin{aligned}
\left\| w_{\alpha,\beta} \Delta^k_{h\varphi}(Q_n, \cdot) \right\|_{L_p(\mathcal{I}_{A',h})} &\ge \left\| w_{\alpha,\beta} \Delta^k_{h\varphi}(Q_n, \cdot) \right\|_{L_p[-1+n^{-2}/2, 1-n^{-2}/2]} \\
&\ge c \left\| w_{\alpha,\beta} \Delta^k_{h\varphi}(Q_n, \cdot) \right\|_p.
\end{aligned} \tag{2.10}$$

It now follows from (2.7) that

$$\Delta^k_{h\varphi(x)}(Q_n, x) - \varphi^k(x) Q_n^{(k)}(x) h^k = \sum_{i=1}^{K} \frac{1}{(2i)!} \varphi^{k+2i}(x) Q_n^{(k+2i)}(x) h^{k+2i} \xi^{2i}_{k+2i},$$

and so, as above,

$$\left\| w_{\alpha,\beta} \left(\Delta^k_{h\varphi}(Q_n, \cdot) - \varphi^k Q_n^{(k)} h^k \right) \right\|_p^q \le 1/2 \cdot h^{kq} \left\| w_{\alpha,\beta} \varphi^k Q_n^{(k)} \right\|_p^q.$$

Therefore,

$$\left\| w_{\alpha,\beta} \Delta^k_{h\varphi}(Q_n, \cdot) \right\|_p^q \ge 1/2 \cdot h^{kq} \left\| w_{\alpha,\beta} \varphi^k Q_n^{(k)} \right\|_p^q,$$

which combined with (2.9) and (2.10) implies (2.5). □

3 The polynomials of best approximation

For $f \in L_p^{\alpha,\beta}$, let $P_n^* = P_n^*(f) \in \mathbb{P}_n$ and $E_n(f)_{w_{\alpha,\beta},p}$ be a polynomial and the degree of its best weighted approximation, respectively, *i.e.,*

$$E_n(f)_{w_{\alpha,\beta},p} := \inf_{p_n \in \mathbb{P}_n} \|w_{\alpha,\beta}(f - p_n)\|_p = \|w_{\alpha,\beta}(f - P_n^*)\|_p.$$

Recall (see [5, Lemma 2.4] and [6, Theorem 1.4]) that, if $\alpha \ge 0$ and $\beta \ge 0$, then, for any $k \in \mathbb{N}$, $0 < p \le \infty$ and $f \in L_p^{\alpha,\beta}$,

$$\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p} \le c \left\| w_{\alpha,\beta} f \right\|_p, \quad t > 0, \tag{3.1}$$

with c depending only on k, α, β and q. Also, for any $0 < \vartheta \le 1$,

$$E_n(f)_{w_{\alpha,\beta},p} \le c\omega_{k,0}^{\varphi}(f, \vartheta n^{-1})_{\alpha,\beta,p}, \quad n \ge k, \tag{3.2}$$

where c depends on ϑ as well as k, α, β and q.

Theorem 3.1. *Let $k \in \mathbb{N}$, $\alpha, \beta \ge 0$, $0 < p \le \infty$ and $f \in L_p^{\alpha,\beta}$. Then, for any $n \in \mathbb{N}$,*

$$n^{-k} \| w_{\alpha,\beta} \varphi^k P_n^{*(k)} \|_p \le c\omega_{k,0}^{\varphi}(P_n^*, t)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}, \quad t \ge \boldsymbol{\varrho} n^{-1}, \tag{3.3}$$

where constants c depend only on k, α, β and q.

Conversely, for $0 < t \le \boldsymbol{\varrho}/k$ and $n := \lfloor \boldsymbol{\varrho}/t \rfloor$,

$$\begin{aligned} \omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p} &\le c \left(\sum_{j=0}^{\infty} \omega_{k,0}^{\varphi}(P_{2^j n}^*, \boldsymbol{\varrho} 2^{-j} n^{-1})_{\alpha,\beta,p}^q \right)^{1/q} \\ &\le c \left(\sum_{j=0}^{\infty} 2^{-jkq} n^{-kq} \| w_{\alpha,\beta} \varphi^k P_{2^j n}^{*(k)} \|_p^q \right)^{1/q}, \end{aligned} \tag{3.4}$$

where c depends only on k, α, β and q.

Corollary 3.2. *Let $k \in \mathbb{N}$, $\alpha, \beta \ge 0$, $0 < p \le \infty$, $f \in L_p^{\alpha,\beta}$ and $\gamma > 0$. Then,*

$$\| w_{\alpha,\beta} \varphi^k P_n^{*(k)} \|_p = O(n^{k-\gamma}) \quad \textit{iff} \quad \omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p} = O(t^{\gamma}). \tag{3.5}$$

Proof of Theorem 3.1. In order to prove (3.3), one may assume that $n \ge k$. By Theorem 2.1 we have

$$n^{-k} \| w_{\alpha,\beta} \varphi^k P_n^{*(k)} \|_p \le c \boldsymbol{\varrho}^{-k} \omega_{k,0}^{\varphi}(P_n^*, \boldsymbol{\varrho} n^{-1})_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(P_n^*, t)_{\alpha,\beta,p}.$$

At the same time, by (3.1) and (3.2) with $\vartheta = \boldsymbol{\varrho}$,

$$\begin{aligned} \omega_{k,0}^{\varphi}(P_n^*, t)_{\alpha,\beta,p}^q &\le \omega_{k,0}^{\varphi}(f - P_n^*, t)_{\alpha,\beta,p}^q + \omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}^q \\ &\le c \| w_{\alpha,\beta}(f - P_n^*) \|_p^q + \omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}^q \\ &\le c\omega_{k,0}^{\varphi}(f, \boldsymbol{\varrho} n^{-1})_{\alpha,\beta,p}^q + \omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}^q \\ &\le c\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}^q, \end{aligned}$$

and (3.3) follows.

In order to prove (3.4) we follow [4]. Assume that $0 < t \le \boldsymbol{\varrho}/k$ and note that $n = \lfloor \boldsymbol{\varrho}/t \rfloor \ge k$. Let $\hat{P}_n \in \mathbb{P}_n$ be a polynomial of best weighted approximation of P_{2n}^*, i.e.,

$$I_n := \left\| w_{\alpha,\beta}(P_{2n}^* - \hat{P}_n) \right\|_p = E_n(P_{2n}^*)_{w_{\alpha,\beta},p}.$$

Then, (3.2) with $\vartheta = \boldsymbol{\varrho}/2$ implies that

$$I_n \le c\omega_{k,0}^{\varphi}(P_{2n}^*, \boldsymbol{\varrho}(2n)^{-1})_{\alpha,\beta,p},$$

while

$$I_n^q \ge \|w_{\alpha,\beta}(f-\hat{P}_n)\|_p^q - \|w_{\alpha,\beta}(f-P_{2n}^*)\|_p^q \ge E_n(f)^q_{w_{\alpha,\beta},p} - E_{2n}(f)^q_{w_{\alpha,\beta},p}.$$

Combining the above inequalities we obtain

$$\begin{aligned} E_n(f)^q_{w_{\alpha,\beta},p} &= \sum_{j=0}^{\infty}\big(E_{2^j n}(f)^q_{w_{\alpha,\beta},p} - E_{2^{j+1}n}(f)^q_{w_{\alpha,\beta},p}\big) \le \sum_{j=0}^{\infty} I^q_{2^j n} \\ &\le c\sum_{j=1}^{\infty}\omega^{\varphi}_{k,0}(P^*_{2^j n}, \boldsymbol{\varrho}2^{-j}n^{-1})^q_{\alpha,\beta,p}. \end{aligned}$$

Hence,

$$\begin{aligned} \omega^{\varphi}_{k,0}(f,t)^q_{\alpha,\beta,p} &\le c\omega^{\varphi}_{k,0}(f-P_n^*,t)^q_{\alpha,\beta,p} + c\omega^{\varphi}_{k,0}(P_n^*,t)^q_{\alpha,\beta,p} \\ &\le cE_n(f)^q_{w_{\alpha,\beta},p} + c\omega^{\varphi}_{k,0}(P_n^*,\boldsymbol{\varrho}n^{-1})^q_{\alpha,\beta,p} \\ &\le c\sum_{j=0}^{\infty}\omega^{\varphi}_{k,0}(P^*_{2^j n},\boldsymbol{\varrho}2^{-j}n^{-1})^q_{\alpha,\beta,p} \\ &\le c\sum_{j=0}^{\infty}2^{-jkq}n^{-kq}\|w_{\alpha,\beta}\varphi^k P^{*(k)}_{2^j n}\|_p^q, \end{aligned}$$

where, for the last inequality, we used Theorem 2.1. This completes the proof of (3.4). □

4 Further properties of the moduli

Following [5, Definition 1.4], for $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, $1 \le p \le \infty$, we define the weighted K-functional as follows

$$\begin{aligned} &K^{\varphi}_{k,r}(f^{(r)},t^k)_{\alpha,\beta,p} \\ &\quad := \inf_{g\in\mathbb{B}_p^{k+r}(w_{\alpha,\beta})}\left\{\left\|w_{\alpha,\beta}\varphi^r(f^{(r)}-g^{(r)})\right\|_p + t^k\left\|w_{\alpha,\beta}\varphi^{k+r}g^{(k+r)}\right\|_p\right\}. \end{aligned}$$

We note that

$$K_{k,\varphi}(f,t^k)_{w_{\alpha,\beta},p} = K^{\varphi}_{k,0}(f,t^k)_{\alpha,\beta,p},$$

where $K_{k,\varphi}(f,t^k)_{w,p}$ is the weighted K-functional that was defined in [3, p. 55 (6.1.1)] as

$$K_{k,\varphi}(f,t^k)_{w,p} := \inf_{g\in\mathbb{B}_p^k(w)}\{\|w(f-g)\|_p + t^k\|w\varphi^k g^{(k)}\|_p\}.$$

The following lemma immediately follows from [5, Corollary 1.7].

Lemma 4.1. *If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, $1 \le p \le \infty$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, then, for all $0 < t \le 2/k$,*

$$K^{\varphi}_{k,r}(f^{(r)},t^k)_{\alpha,\beta,p} \le c\omega^{\varphi}_{k,r}(f^{(r)},t)_{\alpha,\beta,p} \le cK^{\varphi}_{k,r}(f^{(r)},t^k)_{\alpha,\beta,p}.$$

Hence,

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p} \sim K_{k,r}^{\varphi}(f^{(r)},t^k)_{\alpha,\beta,p} = K_{k,\varphi}(f^{(r)},t^k)_{w_{\alpha+r/2,\beta+r/2},p}, \tag{4.1}$$

provided that all conditions in Lemma 4.1 are satisfied.

The following sharp Marchaud inequality was proved in [1] for $f \in L_p^{\alpha,\beta}$, $1 < p < \infty$.

Theorem 4.2 ([1, Theorem 7.5])**.** *For $m \in \mathbb{N}$, $1 < p < \infty$ and $\alpha, \beta \in J_p$, we have*

$$K_{m,\varphi}(f,t^m)_{w_{\alpha,\beta},p} \le Ct^m \left(\int_t^1 \frac{K_{m+1,\varphi}(f,u^{m+1})_{w_{\alpha,\beta},p}^{s_*}}{u^{ms_*+1}}\,du + E_m(f)_{w_{\alpha,\beta},p}^{s_*} \right)^{1/s_*}$$

and

$$K_{m,\varphi}(f,t^m)_{w_{\alpha,\beta},p} \le Ct^m \left(\sum_{n<1/t} n^{s_*m-1} E_n(f)_{w_{\alpha,\beta},p}^{s_*} \right)^{1/s_*},$$

where $s_* = \min\{2,p\}$.

In view of (4.1), the following result holds.

Corollary 4.3. *For $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$, $r/2 + \alpha \ge 0$, $r/2 + \beta \ge 0$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, we have*

$$\omega_{m,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p} \le Ct^m \left(\int_t^1 \frac{\omega_{m+1,r}^{\varphi}(f^{(r)},u)_{\alpha,\beta,p}^{s_*}}{u^{ms_*+1}}\,du + E_m(f^{(r)})_{w_{\alpha,\beta}\varphi^r,p}^{s_*} \right)^{1/s_*}$$

and

$$\omega_{m,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p} \le Ct^m \left(\sum_{n<1/t} n^{s_*m-1} E_n(f^{(r)})_{w_{\alpha,\beta}\varphi^r,p}^{s_*} \right)^{1/s_*},$$

where $s_* = \min\{2,p\}$.

The following sharp Jackson inequality was proved in [2].

Theorem 4.4 ([2, Theorem 6.2])**.** *For $1 < p < \infty$, $\alpha, \beta \in J_p$ and $m \in \mathbb{N}$, we have*

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs^*} E_{2^j}(f)_{w_{\alpha,\beta},p}^{s^*} \right)^{1/s^*} \le CK_{m,\varphi}(f,2^{-nm})_{w_{\alpha,\beta},p}$$

and

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs^*} K_{m+1,\varphi}(f,2^{-j(m+1)})_{w_{\alpha,\beta},p}^{s^*} \right)^{1/s^*} \le CK_{m,\varphi}(f,2^{-nm})_{w_{\alpha,\beta},p},$$

where $2^{j_0} \ge m$ and $s^* = \max\{p,2\}$.

Again, by virtue of (4.1), we have,

Corollary 4.5. *For $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$, $r/2 + \alpha \geq 0$, $r/2 + \beta \geq 0$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, we have*

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs^*} E_{2^j}(f^{(r)})_{w_{\alpha,\beta}\varphi^r,p}^{s^*} \right)^{1/s^*} \leq C\omega_{m,r}^{\varphi}(f^{(r)}, 2^{-n})_{\alpha,\beta,p}$$

and

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs^*} \omega_{m+1,r}^{\varphi}(f^{(r)}, 2^{-j})_{\alpha,\beta,p}^{s^*} \right)^{1/s^*} \leq C\omega_{m,r}^{\varphi}(f^{(r)}, 2^{-n})_{\alpha,\beta,p},$$

where $2^{j_0} \geq m$ and $s^* = \max\{p, 2\}$.

Corollary 4.6. *For $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$, $r/2 + \alpha \geq 0$, $r/2 + \beta \geq 0$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, we have*

$$t^m \left(\int_t^{1/m} \frac{\omega_{m+1,r}^{\varphi}(f^{(r)}, u)_{\alpha,\beta,p}^{s^*}}{u^{ms^*+1}} \, du \right)^{1/s^*} \leq C\omega_{m,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p}, \quad 0 < t \leq 1/m,$$

where $s^* = \max\{p, 2\}$.

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